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THE INSTABILITY OF THE STATES OF REST OF A VISCOUS IDEALLY CONDUCTING COMPRESSIBLE MEDIUM CONTAINING A MAGNETIC FIELD†

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The problem of the linear stability of the states of rest of a viscous compressible medium with infinite conductivity in a magnetic field is investigated. It is shown, using Lyapunov's direct method, that these states of rest are unstable to small spatial perturbations, which reduce the effective potential energy, that is the sum of the internal energy of the medium and the magnetic-field energy. A priori bilateral exponential estimates of the increase in the perturbations are obtained, where the exponents in these estimates are calculated from the parameters of the state of rest and the initial data for the perturbations. A class of the most rapidly growing perturbations is obtained and an exact formula for determining their growth rate is derived. An example of the states of rest and of the initial perturbations which evolve in accordance with the estimates obtained so long as the linear approximation holds is constructed. © 2000 Elsevier Science Ltd. All rights reserved.

The present paper is a natural development and extension of well-known results [1, 2] on new magnetohydrodynamic equilibria of a plasma, since it simultaneously takes into account such physical properties of the medium as the viscosity and compressibility.

1. FORMULATION OF THE EXACT PROBLEM

The three-dimensional motions of a viscous compressible medium of infinite conductivity situated in a magnetic field are investigated. It is assumed that the region τ in which the medium flows is bounded by fixed impenetrable ideally conducting walls $\partial\tau$. The following notation is used below: ρ, p, T, s, e and $\mathbf{v} = (v_1, v_2, v_3)$ are the density, pressure, temperature, entropy, internal energy and velocity fields, $\mathbf{h} = (h_1, h_2, h_3)$ is the magnetic field, η and ζ are constant positive coefficients of the dynamic and second viscosity, $\mathbf{x} = (x_1, x_2, x_3)$ are Cartesian coordinates and t is the time. In this notation the equations of motion of the medium [3, 4] take the form

$$\rho\frac{dv_i}{dt} = \sigma_{ik,k} + \frac{h_k}{4\pi}(h_{i,k} - h_{k,i}), \quad h_{i,i} = 0$$
$$\frac{d\rho}{dt} + \rho v_{i,i} = 0, \quad \frac{dh_i}{dt} = h_k v_{i,k} - h_i v_{k,k}, \quad \rho T\frac{ds}{dt} = D_{ik}v_{i,k} \tag{1.1}$$
$$e = e(\rho, s): de = Tds + \frac{p}{\rho^2}d\rho$$

Here

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + v_k\frac{\partial}{\partial x_k}, \quad \sigma_{ik} \equiv -p\delta_{ik} + D_{ik}$$
$$D_{ik} \equiv \eta(v_{i,k} + v_{k,i} - \frac{2}{3}\delta_{ik}v_{l,l}) + \zeta\delta_{ik}v_{l,l}$$

It is assumed that the following conditions are satisfied on the boundary

$$v_i = 0, \quad h_i n_i = 0 \tag{1.2}$$

where $\mathbf{n} = (n_1, n_2, n_3)$ is the unit outward normal to the boundary $\partial\tau$ of the region in which the medium flows. Everywhere summation is carried out over repeated vector and tensor indices from 1 to 3.

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The initial data for boundary-value problem (1.1), (1.2) are taken in the form

$$\mathbf{v}(\mathbf{x},\ 0)=\mathbf{v}^0(\mathbf{x}),\quad \rho(\mathbf{x},\ 0)=\rho^0(\mathbf{x}),\quad \mathbf{h}(\mathbf{x},\ 0)=\mathbf{h}^0(\mathbf{x}),\quad s(\mathbf{x},\ 0)=s^0(\mathbf{x}) \tag{1.3}$$

where the functions $\rho^0(\mathbf{x})$ and $s^0(\mathbf{x})$ should not contradict the equation of state of the medium, while the function $\mathbf{h}^0(\mathbf{x})$ must convert the second of the equations of system (1.1) into an identity and, together with the function $\mathbf{v}^0(\mathbf{x})$, must agree with boundary conditions (1.2).

We will assume that all the fields present in relations (1.1)–(1.3) possess the necessary degree of smoothness.

The exact non-stationary solutions of initial-boundary-value problem (1.1)–(1.3) must obey the law of conservation of energy

$$E_1 = K_1 + \Pi_1 = \text{const}$$

Here

$$2K_1 \equiv \int \rho v_i v_i d\tau,\quad d\tau \equiv dx_1 dx_2 dx_3,\quad \Pi_1 \equiv \int [\rho e(\rho,\ s)+(8\pi)^{-1} h_i h_i] d\tau$$

(integration is carried out everywhere over the region τ in which the medium flows).

The exact stationary solutions of mixed problem (1.1)–(1.3)

$$\mathbf{v}=\mathbf{v}_0(\mathbf{x})\equiv 0,\quad \rho=\rho_0(\mathbf{x}),\quad p=p_0(\mathbf{x}),\quad \mathbf{h}=\mathbf{h}_0(\mathbf{x}),\quad s=s_0(\mathbf{x}) \tag{1.4}$$

which correspond to states of equilibrium (rest) of a viscous ideally conducting compressible medium in a magnetic field, satisfy the equations.

$$\frac{h_{0k}}{4\pi}(h_{0i,k}-h_{0k,i})=p_{0,i},\quad h_{0i,i}=0,\quad e=e_0(\rho_0,\ s_0) \tag{1.5}$$

(where $p_0 \equiv \rho_0^2 e_\rho(\rho_0, s_0)$) in the region τ and the condition

$$h_{0i}n_i=0 \tag{1.6}$$

on its boundary $\partial\tau$.

2. FORMULATION OF THE LINEARIZED PROBLEM

We linearize initial-boundary-value problem (1.1)–(1.3) on the stationary solutions (1.4)–(1.6), as a result of which we obtain the following system of equations

$$\begin{gathered}\rho_0 v'_{it}=\sigma'_{ik,k}+\frac{h'_k}{4\pi}(h_{0i,k}-h_{0k,i})+\frac{h_{0k}}{4\pi}(h'_{i,k}-h'_{k,i})\\ h'_{i,i}=0,\quad \rho'_t+v'_i\rho_{0,i}+\rho_0 v'_{i,i}=0\\ h'_{it}+v'_k h_{0i,k}=h_{0k}v'_{i,k}-h_{0i}v'_{k,k},\quad s'_t+v'_i s_{0,i}=0\end{gathered} \tag{2.1}$$

which determine the development with time of small perturbations of the velocity field $\mathbf{v}'$, the density field ρ', the entropy field s' and the magnetic field $\mathbf{h}'$ in the region τ in which the medium flows (here a partial derivative with respect to time is denoted by a Latin subscript t). We add to this system the conditions

$$v'_i=0,\quad h'_i n_i=0 \tag{2.2}$$

which are specified on the boundary $\partial\tau$, and the initial data

$$\mathbf{v}'(\mathbf{x},\ 0)=\mathbf{v}'^0(\mathbf{x}),\ \rho'(\mathbf{x},\ 0)=\rho'^0(\mathbf{x}),\ \mathbf{h}'(\mathbf{x},\ 0)=\mathbf{h}'^0(\mathbf{x}),\ s'(\mathbf{x},\ 0)=s'^0(\mathbf{x}) \tag{2.3}$$

where the functions on the right-hand sides of relations (2.3) are subject to the same limitations as the functions corresponding to them on the right-hand sides of Eqs (1.3).

It should be noted that the tensor σ'_{ik} is identical with the tensor σ'_{ik} when p in the latter is replaced by

$$p' = c_0^2\rho' + \rho_0^2 s' e_{\rho s}(\rho_0,\ s_0)\ (c_0^2 \equiv \rho_0[2e_\rho(\rho_0,\ s_0) + \rho_0 e_{\rho\rho}(\rho_0,\ s_0)])$$

and D_{ik} by D'_{ik}. In turn, the tensor D'_{ik} is converted into the tensor D_{ik} if we replace $\mathbf{v}'$ by $\mathbf{v}$ in it. The primes on the perturbation fields and on the tensor fields σ'_{ik} and D'_{ik} will henceforth be omitted.

In order to demonstrate the instability of states of rest (1.4)–(1.6) to the small spatial perturbations (2.1)–(2.3), it is sufficient to obtain at least one perturbation which grows exponentially with time. This can be done most simply by reducing the region in which a search is made for such a perturbation by considering the motion of a medium consisting exclusively of displacements of fluid particles from their equilibrium positions. Such motions are most simply described using the field of Lagrange displacements $\mathbf{f}$ [5], which satisfies the relation

$$\mathbf{f}(\mathbf{x},\ t) = (f_1,\ f_2,\ f_3) : f_{it} = v_i \tag{2.4}$$

Taking into account the definition of the field $\mathbf{f}$, the linearized mixed problem (2.1)–(2.3) can be written in the form

$$\begin{aligned}
&\rho_0 f_{itt} = -p_{,i} + \frac{h_k}{4\pi}(h_{0i,k} - h_{0k,i}) + \frac{h_{0k}}{4\pi}(h_{i,k} - h_{k,i}) + G_{ikt} \\
&\rho + f_i\rho_{0,i} + \rho_0 f_{i,i} = 0,\quad h_i + f_k h_{0i,k} = h_{0k} f_{i,k} - h_{0i} f_{k,k} \\
&s + f_i s_{0,i} = 0\ \left(G_{ik} \equiv \eta\left(f_{i,k} + f_{k,i} - \frac{2}{3}\delta_{ik} f_{l,l}\right) + \zeta\delta_{ik} f_{l,l}\right) \text{ in } \tau \\
&f_i = 0,\quad h_i n_i = 0 \quad \text{on } \partial\tau \\
&\mathbf{f}(\mathbf{x},\ 0) = \mathbf{f}^0(\mathbf{x}),\quad \mathbf{v}(\mathbf{x},\ 0) = \mathbf{v}^0(\mathbf{x})
\end{aligned} \tag{2.5}$$

The energy dissipation equation

$$\dot{E} = -D \tag{2.6}$$

holds on the solutions of initial-boundary-value problem (2.4), (2.5) where

$$E \equiv K + \Pi,\ 2K \equiv \int \rho_0 v_i v_i d\tau,\ \ 2\Pi \equiv \int\left[-pf_{k,k} + \frac{h_i}{4\pi}(h_i - f_k[h_{0k,i} - h_{0i,k}])\right]d\tau$$

$$D \equiv \int\left[\frac{\eta}{2}(v_{i,k} + v_{k,i} - \frac{2}{3}\delta_{ik} v_{l,l})^2 + \zeta v_{i,i}^2\right]d\tau$$

(the dot above the letter E denotes a total derivative with respect to time).

We will assume that an initial field of Lagrange displacements $\mathbf{f}^{0*}$ (x) exists, which ensures that the following condition is satisfied

$$\Pi(0) < 0 \tag{2.7}$$

Of course, for other initial fields $\mathbf{f}^0(\mathbf{x})$ (2.5) the inequality sign in (2.7) may be reversed.

Assuming condition (2.7), we will obtain below bilateral estimates which confirm that small perturbations of the states of equilibrium (1.4)–(1.6) grow exponentially with time.

3. THE LYAPUNOV FUNCTIONAL

For this purpose, following the recommendations made in [1, 2], we will introduce auxiliary functions of the form

$$M \equiv \int \rho_0 f_i f_i d\tau,\quad X \equiv \dot{M} + G,\quad G \equiv \int\left[\frac{\eta}{2}(f_{i,k} + f_{k,i} - \frac{2}{3}\delta_{ik} f_{l,l})^2 + \zeta f_{i,i}^2\right]d\tau \tag{3.1}$$

Differentiating the integral X with respect to time and reducing it further using (2.4)–(2.6) and (3.1) we obtain the relation

$$\dot{X} = 4(K - \Pi) = 8K - 4E$$

which is called the virial equation [5]. Multiplying this equation by an arbitrary constant factor λ and subtracting the result obtained from Eq. (2.6), we obtain the relation

$$\dot{E}_\lambda = 2\lambda E_\lambda - 4\lambda K_\lambda - D_\lambda \tag{3.2}$$

Here

$$E_\lambda \equiv K_\lambda + \Pi_\lambda, \quad 2K_\lambda \equiv 2K - \lambda\dot{M} + \lambda^2 M = \int \rho_0(\mathbf{f}_t - \lambda\mathbf{f})^2 d\tau$$

$$2\Pi_\lambda \equiv 2\Pi + \lambda G + \lambda^2 M, \quad D_\lambda \equiv D - \lambda\dot{G} + \lambda^2 G =$$

$$= \int\left[\frac{\eta}{2}(v_{i,k} + v_{k,i} - \frac{2}{3}\delta_{ik}v_{l,l} - \lambda(f_{i,k} + f_{k,i} - \frac{2}{3}\delta_{ik}f_{l,l}))^2 + \zeta(v_{i,i} - \lambda f_{i,i})^2\right]d\tau$$

If we now put $\lambda > 0$, then, in view of the fact that the functionals K_λ and D_λ are non-negative, we obtain the differential inequality $\dot{E}_\lambda \leqslant 2\lambda E_\lambda$, integration of which enables us to establish the relation

$$E_\lambda(t) \leqslant E_\lambda^0 \exp(2\lambda t) (E_\lambda^0 \equiv E_\lambda(0)) \tag{3.3}$$

which plays a decisive role in all the subsequent discussions.

It is important that inequality (3.3) holds for any solutions of mixed problem (2.4), (2.5). Moreover, when deriving it it was not necessary to impose any constraints on the sign of the functional Π (2.6).

Relation (3.3) shows that the integral E_λ (3.2) varies monotonically with time, and it can therefore be considered below as the Lyapunov functional [1, 2].

4. LOWER AND UPPER LIMITS

Suppose condition (2.7) is satisfied. Since the field of Lagrange displacements **f** and the perturbations of the velocity field **v** are specified at the initial instant of time independently of one another, we can take as the latter the function $\mathbf{v}^0$ (2.3), such that the inequality $K^0 < |\Pi^0|$ is true. In this case the integral E_λ^0 (3.2) becomes a second-degree polynomial in λ with a positive coefficient M^0 (3.1) for λ^2 and a negative free term E^0 (2.6)

$$E_\lambda^0 = M^0\lambda^2 + A^0\lambda + E^0 (2A \equiv G - \dot{M}) \tag{4.1}$$

If we choose $\lambda > 0$, it follows from (4.1) that in the interval

$$0 < \lambda < \Lambda_1 = -\frac{A^0}{2M^0} + \sqrt{\left(\frac{A^0}{2M^0}\right)^2 - \frac{E^0}{M^0}} \tag{4.2}$$

the following relation will hold

$$E_\lambda^0 < 0 \tag{4.3}$$

Inequalities (3.3) and (4.3) confirm that the solutions of the initial-boundary-value problem (2.4), (2.5) will increase exponentially with time.

Putting $\lambda = \Lambda_1 - \delta \equiv \sigma$ (with any δ from the interval $[0, \Lambda_1]$) we can give relation (3.3) the form

$$E_\sigma(t) \leqslant E_\sigma^0 \exp[2\sigma t] \ (E_\sigma^0 < 0) \tag{4.4}$$

By the definition of the functional E_λ we have the inequality

$$E_\lambda(t) > \Pi(t)$$

which enables us to rewrite relation (4.4) in the form

$$\Pi(t) < E_\sigma^0 \exp[2\sigma t] \tag{4.5}$$

Using the additional integral

$$J(t) \equiv \int [\rho^2 + s^2 + f_{i,i}^2 + h_i h_i + f_i f_i] d\tau$$

inequality (4.5) is converted to the final and most obvious form

$$J(t) > | c E_\sigma^0 | \exp[2\sigma t] \tag{4.6}$$

where c is a known constant. It follows from (4.6) that the parameter σ (4.2), (4.4) is a lower limit of the increment of the solutions of mixed problem (2.4), (2.5).

Limit (4.6) can be improved considerably if the initial perturbations of the velocity field $\mathbf{v}^0$ is related to the initial field of Lagrange displacements $\mathbf{f}^{0*}$ (2.5), (2.7) by the equality

$$\mathbf{v}^0(\mathbf{x}) = \lambda \mathbf{f}^{0*}(\mathbf{x}) \tag{4.7}$$

In fact, the presence of relation (4.7) is the reason that we can obtain from (3.2) the relations

$$K_\lambda^0 = 0, \quad E_\lambda^0 = \Pi_\lambda^0$$

which, when $\lambda > 0$, and taking (4.1) into account, enables us to be certain of the truth of the inequality $\Pi_\lambda^0 < 0$ in the interval

$$0 < \lambda < \Lambda = -\frac{G^0}{2M^0} + \sqrt{\left(\frac{G^0}{2M^0}\right)^2 - \frac{2\Pi^0}{M^0}} \tag{4.8}$$

If we take $\lambda = \Lambda - \delta_1 \equiv \sigma_1$(with arbitrary δ from the interval $[0, \Lambda]$), we can write relation (3.3) in the form

$$E_{\sigma_1}(t) \leqslant \Pi_{\sigma_1}^0 \exp[2\sigma_1 t] \ (\Pi_{\sigma_1}^0 < 0) \tag{4.9}$$

whence, using the definition of the functional $J(t)$ we can also derive the required improved (i.e. more rigorous) limit

$$J(t) > | c_1 \Pi_{\sigma_1}^0 | \exp[2\sigma_1 t] \tag{4.10}$$

(here c_1 is a known constant quantity). Inequality (4.10) indicates that the parameter σ_1 (4.8), (4.9) gives a lower limit for the values of the increments of the solutions of initial-boundary-value problem (2.4), (2.5), (4.7). A comparison of the limits (4.6) and (4.10) shows that the solutions of mixed problem (2.4), (2.5), the initial data of which are additionally subject to condition (4.7), increase faster than all the remaining perturbations.

However, the greatest increase in the solutions of initial-boundary-value problem (2.4), (2.5) is observed when their growth increment is equal to the following quantity

$$\Lambda^+ \equiv \sup_{\mathbf{f}^{0*}(\mathbf{x})} \Lambda \tag{4.11}$$

In order to prove this assertion we need to obtain the limit from which the upper bound of the growth of small spatial perturbations of states of rest (1.4)–(1.6) follow. To do this the parameter λ is chosen to be strictly greater than Λ^+. Then the integral Π_λ^0 will be positive definite for any possible specification of the initial field of the Lagrange displacements $\mathbf{f}^0$ (2.5). This means that the functional E_λ^0 will also be positive for all possible initial fields of the Lagrange displacements $\mathbf{f}^0$ and perturbations of the velocity field $\mathbf{v}^0$ (2.4), (2.5).

Consequently, taking $\lambda = \Lambda^+ + \varepsilon \equiv \sigma_2$, where $\varepsilon > 0$, we can represent inequality (3.3) in the form

$$E_{\sigma_2} \leqslant E_{\sigma_2}^0 \exp[2\sigma_2 t] \tag{4.12}$$

If we now bear in mind the limit

$$\Pi_{\Lambda^+}(t) \geqslant 0$$

inequality (4.12) can be rewritten in the clearer form

$$2K_{\sigma_2}(t) + \varepsilon(\Lambda^+ + \sigma_2)M(t) + \varepsilon G(t) \leqslant 2E_{\sigma_2}^0 \exp[2\sigma_2 t] \tag{4.13}$$

Relation (4.13) confirms that the parameter σ_2 (4.8), (4.11) gives an upper limit of the increment of the solutions of mixed problem (2.4), (2.5).

A comparison of the limits (4.10) and (4.13) enables us to conclude that the parameter Λ^+ sets both a lower and upper limit to the growth of the solutions of initial-boundary-value problem (2.4), (2.5)

$$\Lambda^+ - \delta_1 \leqslant \omega_* \leqslant \Lambda^+ + \varepsilon \tag{4.14}$$

In turn, the double inequality (4.14) enables us to conclude that the solutions of mixed problem (2.4), (2.5) that grow most rapidly are those for which the increment is identical in value with Λ^+ (4.11).

Thus, if condition (2.7) is satisfied, then, by calculating the value of Λ^+ from formulae (4.10) and (4.11) we can determine the characteristic time during which a viscous ideally conducting compressible medium, containing a magnetic field, transfers to some state of equilibrium (1.4)–(1.6).

5. EXAMPLE

Consider, in a cylindrical system of coordinates r, φ, z, the magneto-hydrodynamic states of rest of a viscous compressible medium of infinite conductivity, which completely fills the space between two coaxial cylinders, limited in height

$$\begin{gathered}\mathbf{v}_0 \equiv 0, \quad \mathbf{h}_0 = (0,\ \alpha r^2,\ 0), \quad p_0 = \frac{3\alpha^2}{16\pi}[r_2^4 - r^4] \\ e_0 = e_0(\rho_0,\ s_0), \quad \rho_0 = \rho_0(r), \quad s_0 = s_0(r) \\ \tau = [(r,\ z) : 0 < r_1 < r < r_2,\ \ 0 < z < z_1]\end{gathered} \tag{5.1}$$

(here α, r_1, r_2 and z_1 are constant quantities). It should be noted that the equation of state of the medium is assumed to be arbitrary, while the functions ρ_0 and s_0 are chosen so as to obtain the required distribution of the pressure p_0 for any equation of states specified in advance.

The states of equilibrium (5.1), which can be verified by direct calculations, are the exact solutions of stationary equations (1.5) in the region τ whereas on its surface $\partial\tau = [(r, z) : r = r_1, r = r_2, z = 0, z = z_1]$ the magnetic field h_0 satisfies boundary condition (1.6).

It turns out that these states of equilibrium will be unstable, for example, to those small perturbations to which the initial Lagrange displacement field corresponds

$$\mathbf{f}^{0*} = \left(r^{-1}\frac{\partial q_2}{\partial z},\ q_1,\ -r^{-1}\frac{\partial q_2}{\partial r}\right) \tag{5.2}$$

where it is required of the function $q_1(r, z)$ and $q_2(r, z)$ only that they ensure the validity of boundary conditions (2.5). Otherwise, these functions are assumed to be quite arbitrary and, in particular, can be taken in the following form

$$q_k = a_k(r - r_1)^k (r - r_2)^k z^k (z - z_1)^k (k = 1, 2) \tag{5.3}$$

where a_1 and a_2 are certain constants.

In fact, by making the necessary calculations it is easy to show that in this case

$$\Pi(0) = -\frac{\alpha^2}{2}\int_{r_1}^{r_2}\int_0^{z_1}\left(\frac{\partial q_1}{\partial z}\right)^2 r dr dz < 0$$

i.e. condition (2.7) holds. Hence, perturbations (5.2) and (5.3) will grow with time in accordance with the limits (4.6), (4.10) and (4.13), and there rate of growth will be given by formulae (4.2), (4.8) and (4.11).

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